Structural Analysis of Masonry Historical Constructions. Classical and Advanced Approaches

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Abstract

A review of methods applicable to the study of masonry historical construction, encompassing both classical and advanced ones, is presented. Firstly, the paper offers a discussion on the main challenges posed by historical structures and the desirable conditions that approaches oriented to the modeling and analysis of this type of structures should accomplish. Secondly, the main available methods which are actually used for study masonry historical structures are referred to and discussed.

The main available strategies, including limit analysis, simplified methods, FEM macro- or micro-modeling and discrete element methods (DEM) are considered with regard to their realism, computer efficiency, data availability and real applicability to large structures. A set of final considerations are offered on the real possibility of carrying out realistic analysis of complex historic masonry structures. In spite of the modern developments, the study of historical buildings is still facing significant difficulties linked to computational effort, possibility of input data acquisition and limited realism of methods.

Keywords

Historical construction, structural analysis, masonry mechanics, limit analysis, macro-modeling, micro-modeling, discontinuous methods.

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1. Introduction. Purpose and challenges

Studies oriented to conservation and restoration of historical structures have recourse to structural analysis as a way to better understand the genuine structural features of the building, to characterize its present condition and actual causes of existing damage, to determine the true structural safety for a variety of actions (such as gravity, soil settlements, wind and earthquake) and to conclude on necessary remedial measures. In short, structural analysis contributes to all the phases and activities (including diagnosis, reliability assessment and design of intervention) oriented to grant an efficient and respectful conservation of monuments and historical buildings. Accurate structural analysis is needed to avoid erroneous or defective conclusions leading to either over-strengthen the structure, causing unnecessary loss in terms of original material and cultural value, or to insufficiently intervene on it, and hence generate inadmissible risks on people and heritage. Unsurprisingly, ancient structures have been studied, since long time ago, using the most advanced tools available for structural assessment.

The application of advanced computer methods to the analysis of historical structures was pioneered by the studies of the Brunelleschi Dome by Chiarugi et al. (1993), the Pisa Tower by Macchi et al. (1993), the Colosseo in Rome by Croci (1995), see also Croci and Viskovic (1993), Mexico Cathedral by Meli and Sánchez-Ramírez (1995) and San Marco's Basilica in Venice by Mola and Vitaliani (1995), among others (Figs.1 and 2). By then, the development of methods for accurate analysis of steel and concrete structures, including non-linear applications, was already at a very advanced stage thanks to the work of Zienkiewicz and Taylor (1991), Ngo and Scordelis (1964) and many others. Notwithstanding, analysts attempting to use computer tools for the study historical structures were by then facing overwhelming challenges. Methods then available were not yet prepared to tackle the specific problems of ancient constructions concerning materials, structural arrangements and real preservation condition. In fact, the difficulties posed by historical structures are still very challenging, and still reminiscent of those encountered by the pioneers, in spite of significant progress during the last decades.

Some of difficulties encountered are related to the description of geometry, materials and actions, all of which acquire remarkable singularity in the case of historical construction. Additional important difficulties are related to the acquisition of data on material properties, internal morphology and damage, as well as to the adequate interpretation of structural arrangements, overall organization and historical facts. Because of all these difficulties, it is generally accepted (Icomos/Iscarsah Committee, 2005) that the study of a historical structure should not only base on calculations, but should integrate as well a variety of complementary activities involving detailed historical investigation, deep inspection by means of non destructive techniques (NDT) and monitoring, among other. Structural analysis of historical structures constitutes in fact a multidisciplinary, multifaceted activity requiring a clever integration of different approaches and sources of evidence. These difficulties are discussed into more detail in the following paragraphs.

1- Material. Historical or traditional materials such as earth, brick or stone masonry and wood are characterized by very complex mechanical and strength phenomena still challenging our modeling abilities. In particular, masonry is characterized by its composite character (it includes stone or brick in combination with mortar or day joints), a brittle response in tension (with almost null tensile strength), a frictional response in shear (once the limited bond between units and mortar is lost) and anisotropy (for the response is highly sensitive to the orientation of loads). In spite of the very significant effort invested to characterize and mathematically describe masonry mechanics and strength, the accurate and efficient simulation of masonry response is still a challenge in need of further experimental and theoretical developments. Important results by Ali and Page (1988), Lourenço (1996), Binda et al. (1997) and many others have yielded a very significant level of understanding.

Historical materials, including brick or stone masonry, are normally very heterogeneous even in a single building or construction member. Moreover, historical structures often show many additions and repairs done with different materials. Material characterization is constrained by due respect to the monument and original material. Non- destructive, indirect, tests (NDT) and minor destructive tests (MDT) should be preferred. If any, only a very limited number of pits or cores allowing direct observation and laboratory testing are normally acceptable. In practice, only limited and partial information can be collected. Additional assumptions on morphology and material properties may be needed in order to elaborate a model.

2- Geometry. Historical structures are often characterized by a very complex geometry. They often include straight or curved members. They combine curved 1D members (arches, flying arches) with 2D members (vaults, domes) and 3D ones (fillings, pendentives…). They combine slender members with massive ones (massive piers, walls buttresses, foundations…). However, today numerical methods (such as FEM) do afford a realistic and accurate description of geometry. Due to it, geometry is perhaps one of the least (although still meaningful) challenges to be faced by the analysis.

3. Morphology. A more significant problem lays in the characterization and description of the internal morphology of structural members and their connections. Structural members are often non-homogeneous and show complex internal structures including several layers, filling material, cavities, metal insertions and other possible singularities. Connections are singular regions featuring specific geometric and morphological treats. The transference of forces may activate specific resisting phenomena (contact problems, friction, eccentric loading). Modeling morphology and connections in detail may be extremely demanding from a computational point of view. Nevertheless, the main difficulty is found in physically characterizing them by means of minor- or non-destructive procedures.

4. Actions. Historical structures may have experienced (and keep on experiencing) actions of very different nature, including the effects of gravity forces in the long term, earthquake, environmental effects (thermal effects, chemical or physical attack), and anthropogenic actions such as architectural alterations, intentional destruction, inadequate restorations… Many of these actions are to be characterized in historical time. Some are cyclic and repetitive (and accumulate significant effect in the long term), some develop gradually in very long time periods, and some are associated to long return periods. In many cases, they may be influenced by historical contingency and uncertain (or at least, insufficiently known) historical facts.

(a)

(b)

(c)

Fig. 1: Some pioneering FEM studies on historical structures. (a) Tower of Pisa: FEM model and substructuring of the colonnade system (Macchi et al, 1993). (b) Mexico City Cathedral (Meli and Sánchez Ramírez, 1995). (c) The Colosseum in Rome. Tensile horizontal stresses due to the seismic action (Croci, 1995).

5. Damage and alterations. Existing and general alterations may affect very significantly the response of the structure. Damage and deformation are to be modeled, as present features of any existing structure, to grant adequate realism and accuracy in the prediction of the actual performance and capacity. Damage encompasses mechanical cracking, material decay (due to chemical or physical attack) or whatever phenomena influencing on the original capacity of materials and structural members.

6. History. History is an essential dimension of the building and must be considered and integrated in the model. The following effects linked to history may have had influence on the structural response and existing damage: Construction process, architectural alterations, additions, destruction in occasion of conflicts (wars…), natural disasters (earthquake, floods, fires…) and long-term decay or damage phenomena. History constitutes a source for knowledge. In many occasions, the historical performance of the building can be engineered to obtain conclusions on the structural performance and strength. For instance, the performance shown during past earthquakes can be considered to improve the understanding on the seismic capacity. The history of the building constitutes a unique experiment occurred in true scale of space and time. In a way, knowledge of historical performance makes up for the mentioned data insufficiency.

2. Desirable features of methods applied to historical structures

Because of the aforementioned challenges, attempts to model and simulate the response of a historical structure should try to satisfy some basic requirements. Firstly, any modelling technique should be able to adequately describe the geometry and morphology of the real construction, including the structural form, internal composition, connections and support conditions. An accurate description of the distribution of mass and external forces is essential for both gravity and seismic analyses.

Secondly, constitutive equations should be adopted allowing an adequate description of the essential mechanical and strength features of the different materials existing in the building. It is important to highlight that simple linear elastic analysis fails to simulate essential features of non-tension resisting materials such as stone and masonry. More sophisticated, non-linear constitutive equations will normally be necessary. In turn, the use of such constitutive equations will require the availability of non-linear properties to be obtained by means of different laboratory or in-situ mechanical tests.

Actions (mechanical, physical, chemical...) are also to be modelled by means of mathematical formulations describing their mechanical effect in terms for forces on the structure, imposed movements or deformations, or possible variations of the material properties.

An accurate model of the structure should also afford the description of damage and alterations existing in the structure, including cracks, disconnections, crushing, deformation and out-of-plumb, and construction defects. Some damage types can be modelled indirectly as a disconnection between elements or a local reduction of material properties. In order to characterize the actual capacity in the present condition of a building, the analysis should be carried out on the model accounting for its real damaged and deformed state.

As the analysis of historical structures will normally be oriented to identify needs for restoration and strengthening, analysis methods should able to incorporate and model possible stabilization, repair or strengthening measures. In some cases, these can be taken into account in an indirect way by adequately modifying material properties, modifying the sectional dimensions or configuration, or by adding forces to represent their mechanical effect.

The interaction of the structure with the soil is also to be taken into account except in cases it is judged to be irrelevant. Taking it into account will often require the inclusion of a large portion of foundation soil as part of the entire FEM model as done in the analysis of San Marcos Basilica by Mola and Vitaliani (1995) or the modal analysis of a masonry tower by Fanelli (1993) (Figs. 2 and 3).

Fig. 2: The finite element model of the St. Mark's Basilica: Top: global discretization; Bottom: soil foundation discretization including deformation (Mola and Vitaliani 1995).

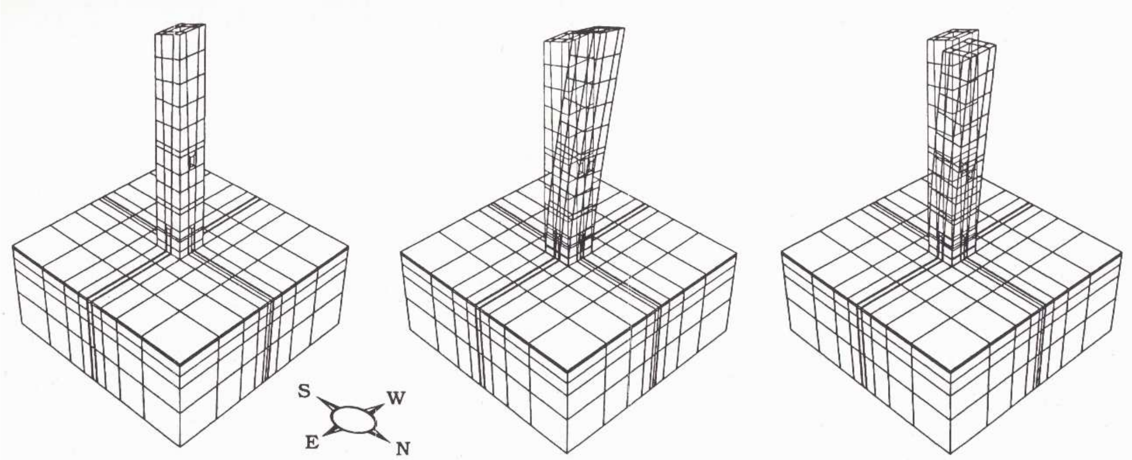
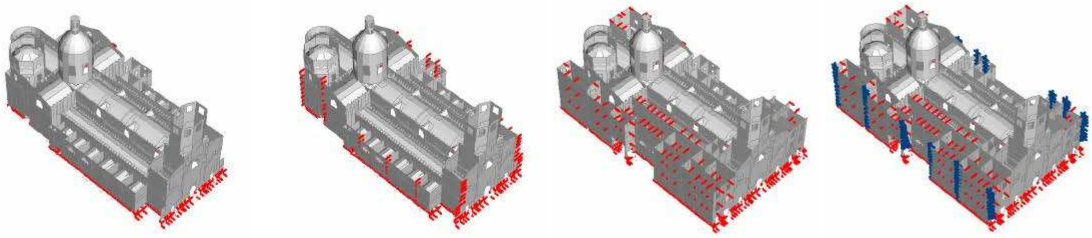


Fig. 3: Examples of graphical rendering for a FE mathematical model of a masonry tower: discretized geometry, including the soil foundation, and first two vibration modal shapes (Fanelli, 1993).

Certain types of analyses, as in particular dynamic one, may require the inclusion of neighbouring buildings into the model with an adequate description of existing connections. This is so because of their possible effect on the modal shapes and overall dynamic response. Modelling accurately the dynamic response will often require to construct a global model incorporating all the distinct parts of a complex structure as in the analysis of Reggio Emilia Cathedral (Casarin, 2006, Casarin and Modena, 2008), of the Monastery of Jerónimos (Lourenço and Mourão, 2001) or of Mallorca Cathedral (Roca et al., 2009) (Fig. 4-6). The connection between the different parts should be modelled accurately by taking into account the real contact conditions, which requires a previous detailed inspection and investigation. The agreement between numerical and experimental vibration modes and frequencies may be considered as a way to validate the description of connection among different parts in the structure. .



a) b) c) d)

Fig. 4: Different stages of the model (a) to (d) and successive boundary conditions variation (Casarin, 2006). The surrounding buildings are taken into account by directly modelling their walls or by simulating them with translational springs.

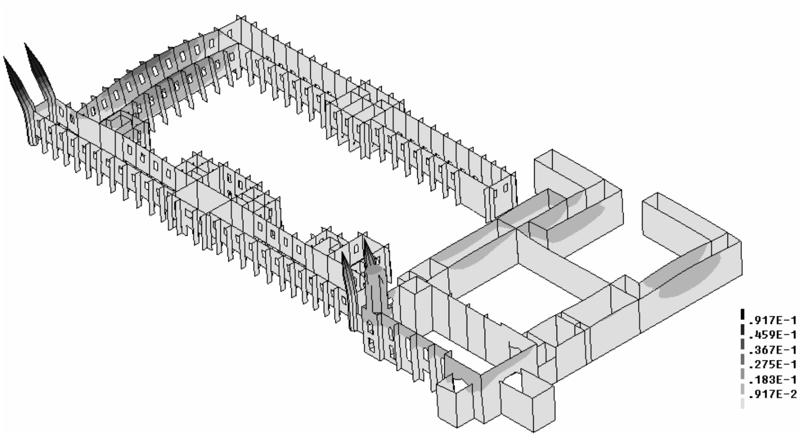


Fig. 5: Deformed meshes and maximum displacements (in grey scale) for seismic load acting along the longitudinal direction of the building. Monastery of Jerónimos (Lourenço and Mourão, 2001).

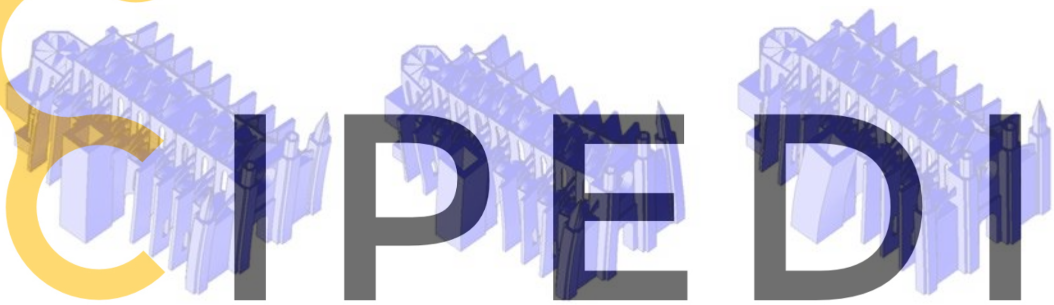


Fig. 6. Detailed modelling of Mallorca cathedral, in Spain, including the different parts of the structure which can influence on the modal analysis (tower, façade, choir) and first modal shapes (Roca et al., 2009).

Studies on different historical structures (Roca 2004, Roca et al., 2004) have shown that real deformations are normally much larger (one or even two orders of magnitude) than those predicted by conventional instantaneous calculations. This is due to the fact that these analysis neglect history-related aspects such as (1) deformations occurred during the construction process, as those due to the deflection of centerings and forms, and the deformation of structural members during intermediate and incomplete configurations of the structure, (2) initial and historical soil settlements, (3) architectural alterations, (4) the non-reversible effect of multiple thermal and hygrometric cycles, and (4) long term damage of mechanical, physical or chemical nature, among other phenomena.

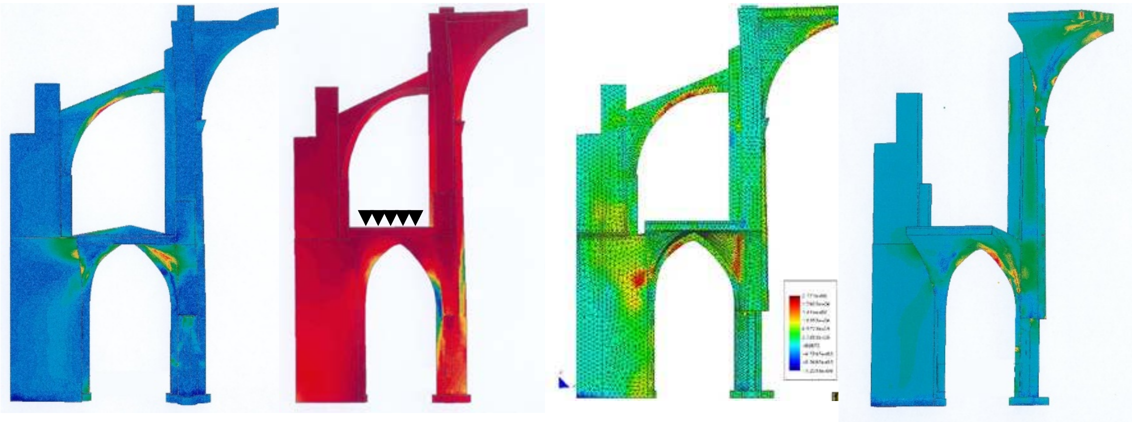
A significant part of the damage and deformation today visible in ancient construction may have been experienced during the construction process or at very early stages of the structure's life. Because of it, it is hardly possible to determine the amount of deformation mechanically connected to the gravity forces or other possible actions. In any case, a good approach to the prediction of deformation and, possibly, damage, can hardly be derived from a purely instantaneous analysis which does not take into account, to some extent, the changes and historical events or physical phenomena affecting the structure. Accurate studies should ideally afford the simulation of the following aspects:

(1) the subsequent historical stages experienced by the building (in particular, the construction process) through a sequential analysis,
(2) the actions occurring in historical periods, such as major earthquakes or the repeated effect of minor earthquakes or thermal cycles, and
(3) long-term damage processes (such as those related to long-term creep) developed across the life of the construction.

As observed apropos of the study of the collapse of the Civic Tower of Pavia and Noto Cathedral in Italy, (Papa and Taliercio, 2000, Binda, et al. 1992, 2001, 2003), the effect of creep under constant stress, at the long term, may induce significant, cumulative damage in rock-like materials. The investigation of specimens cut from the walls of the Pavia Tower after its collapse allowed to Anzani et al. (2008) the formulation, for the first time on ancient masonry, of the hypothesis of a collapse due to the long-term behaviour of the material. The identification of the problem fostered some research effort to better characterize the phenomenon, and some models have been already proposed for its description (Lourenço and Pina-Henriques, 2008, Taliercio and Papa, 2008).

It must be noted, however, that a realistic simulation of historical actions or long-term damage processes is still requiring additional experimental studies and numerical developments. In fact, limitations linked to the capacity of the computers and data available make it very difficult to integrate the aforementioned capabilities in the analyses. A realistic modelling including both structural history and time-dependent effects is, at the moment, still too demanding with regard to the possibilities of conventional calculations approaches. However, these aspects may be taken into account in an approximate way or may at least be considered to obtain a better interpretation the results.

As an attempt to examine the influence of the architectural alterations experienced by the structure of Tarazona Cathedral in Spain (Roca, 2001, Fig. 7), the analysis was independently carried out on a series of structural configurations corresponding to different historical moments. For the analysis of Mallorca cathedral, whose final condition is believed to have been very much affected by a delicate construction process, a true sequential analysis was carried out involving the superposition of two consecutive construction stages (Clemente, 2006, Fig. 8). The construction process was reconstructed thanks to a detailed research on historical documents. The deformation and damage predicted by the sequential analysis was significantly larger than those obtained by instantaneous analysis on the final structural configuration.



(a) (b) (c) (d)

Fig. 7. Analysis of Tarazona Cathedral for different historical configurations (Roca, 2001) by FEM isotropic damage model. (a) Distribution of tensile damage at the initial condition after construction (12^{th} -13^{th} c.); (b) Compression damage after overloading of lateral vaults during late Middle Age; (c) tensile stresses (N/m^2) after "thinning" of piers (16^{th} c.); tensile damage during momentary dismantling of flying arches for restoration purposes (20^{th} c.).

Fig. 8. Sequential analysis of Mallorca Cathedral taking into account the construction process (Clemente 2006) using a FEM isotropic damage model. Phases considered (left) and corresponding distribution of deformation and damage (right).

3. Review of classical methods

It was Robert Hooke who discovered that the ideal shape of a masonry arch in equilibrium is that of the inverted catenary curve drawn by a chain subjected to the same weight distribution. At the end of a printed lecture on *Helioscopes and some other instruments* in 1676 he inserted the following problem: "The true mathematical and mechanical form of all manner of arches for building, with the true butment necessary to each of them. A problem which no architectonick writer hath ever yet attemted, much less performed". He then provided the solution in the form of an anagram whose decipherment was only revealed after his death in 1705. The solution read: "Ut pendet continuum flexile, sic stabit contiguum rigidum inversum" - as hangs a flexible cable, so inverted, stand the touching pieces of an arch (Heyman, 1989). Meanwhile, the equation for the curve of a hanging flexible line or catenary had been already derived by David Gregory, who by 1698 had independently reached and extended Hooke's assertion to the case of material arches with finite thickness. According to Gregory, arches are stable when some catenary can be fitted within its thickness.

The possibility of analysing masonry arches by the analogy between the equilibrium of compressed members with that of funicular models was used during 18th and 19th c., and even at the beginning of 20th c., for the design and assessment of masonry bridges and buildings. A well-known example is the study of the dome of St. Peter in the Vatican by Poleni (1743), Fig. 9. At the beginning of 20th c., architect A. Gaudí used the same principle to design complex structures, such as the Church of Colònia Güell near Barcelona, by means of 3D hanging models (Fig. 10).

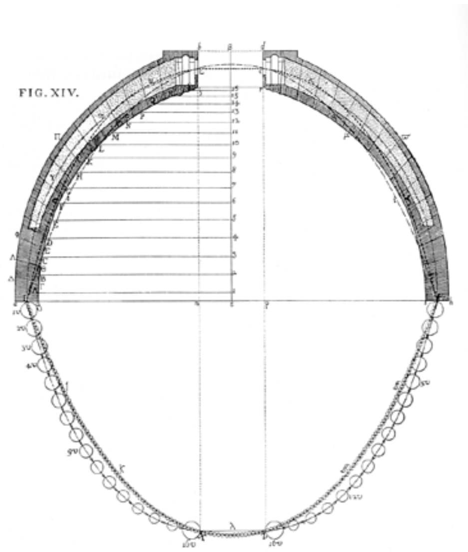


Fig. 9. Poleni's solution to describing the equilibrium of Saint Peter's dome as the inverted shape of a catenary (Poleni, 1743)

Fig. 10. Gaudi's hanging model used to design the church of Colònia Güell c. 1900 (Ràfols 1929).

In France, during the 18th c., La Hire, Couplet and Coulomb undertook the problem from a different approach. Their understanding resulted from viewing the arch as a conjunction of rigid bodies which could experience relative displacements. According to Couplet, the collapse occurs when the arch develops enough hinges (or sections experiencing a relative rotation) as to become a mechanism (Heyman, 1976). The first general theory on the stability of arches was published by Coulomb in his 1773 essay. Coulomb developed a consistent and general theory providing the mathematical base for the description of the different possible modes of collapse, taking into account both relative rotations and sliding between parts. He also stated that the failure due to sliding is rare and suggested to consider only overturning (rotational) failures for practical purposes. He proposed the use of a theory of "maxima and minima" (from our modern point of view, and optimization method) to determine the position of the more unfavourable hinges or sections of rupture.

A further development arrived with the thrust line theory and graphic statics during 19th c. Graphic statics supplied a practical method consistently based on the catenary principle. Graphic statics was actually used for the assessment of a large amount of masonry bridges and large buildings up to the beginning of 20th c. An example is given by Rubio's analysis of the structure of Mallorca Cathedral (Rubió, 1912, Fig. 11). Huerta (2001) provides a comprehensive review of the application of the thrust line theory to the analysis of vaults and domes.

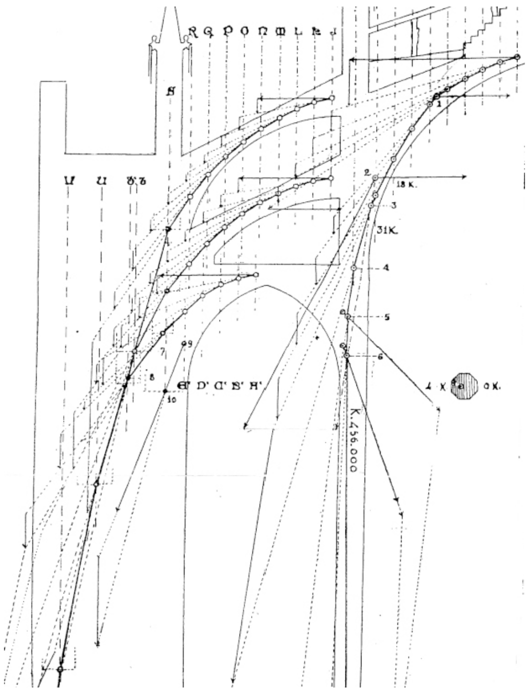


Fig. 11. Rubió's (1912) application of graphic statics to analyze the stability of Mallorca Cathedral (detail).

4. Modern theories to support classical approaches. Plastic analysis.

Heyman's (1966) formulation for the plastic (or limit) analysis of masonry arches synthesizes all the mentioned historical insights and provides a powerful and theoretically sound tool for the study of this type of constructions. According to this formulation, the limit theorems of plasticity can be applied to masonry structures provided the following conditions are verified: (1) The compression strength of the material is infinite; (2) Sliding between parts is impossible; (3) The tensile strength of masonry is null. These conditions enable the application of the well known limit theorems of plasticity.

In particular, these conditions enable the following formulation of the lower-bound (or safe) theorem: The structure is safe, meaning that the collapse will not occur, if a statically admissible state of equilibrium can be found. This occurs when a thrust line can be determined, in equilibrium with the external loads, which falls within the boundaries of the structure. The load applied is a lower-bound of the actual ultimate load (causing failure). The lower-bound theorem supports the so-called static approach (or static limit analysis) for safety assessment of masonry structures.

According to the upper-bound theorem, If a kinematically admissible mechanism can be found, for which the work developed by external forces is positive or zero, then the arch will collapse. In other works, if a mechanism is assumed (by arbitrarily placing a sufficient number hinges), the load which results from equating the work of the external forces to zero is an upper-bound of the actual ultimate load. The application of the upper bound theorem leads to the so-called kinematic approach (or kinematic limit analysis) for the study of masonry buildings.

A corollary of the safe theorem, the so-called uniqueness theorem, is also applicable: A limit condition will be reached (i.e., the structure will be about to collapse) if a both statically and kinematically admissible collapsing mechanism can be found. In other words, the collapsing configuration will be reached if a thrust line can be found causing as many hinges as needed to develop a mechanism. Hinges are caused by the thrust line becoming tangent to the boundaries. When this occurs, the load is the true ultimate load, the mechanism is the true ultimate mechanism, and the thrust line is the only possible one.

In spite of its ancient origin, limit analysis is regarded today as a powerful tool realistically describing the safety and collapse of structures composed by blocks (including not only arches and structures composed of arches, but also towers, façades and entire buildings). It must be remarked, however, that it can hardly be used to describe the response and predict damage for moderate or service load levels not leading to a limit condition. Strictly speaking, limit analysis can only be used to assess the stability or safety of structures.

Limit analysis entails a very deep and conspicuous reality and should be always considered as a complementary tool, or at least as a guiding intuition, when performing alternative computer analyses. Experience shows that, no matter the level of sophistication of any computer method, it will produce, at ultimate condition, results foreseeable by means of limit analysis.

Based on the observation of real seismic failure modes of historical and traditional buildings in Italy, Giuffré (1990, 1995), see also Giuffrè and Carocci (1993) and Carocci (2001), proposed an approach for the study of the seismic vulnerability of masonry buildings based on their decomposition into rigid blocks (Figs. 12 and 13). The collapse mechanisms are then analyzed by applying kinematic limit analysis. This approach is particularly interesting as a tool for seismic analysis of buildings which do not conform to box behaviour because of lack of stiff floor slabs or because of weaker partial collapses affecting the façade or inner walls.

More recently, Giuffré's proposal has experienced renewed interest thanks to the possibility of combining block analysis with the capacity spectrum method (Fajfar 1999, Lagormarsino et al., 2003, Lagomarsino, 2006) for the seismic assessment of masonry structures. The method is applied to buildings, churches and towers (Fig. 14). The resulting verification methodology has been adopted by the seismic Italian code OPCM 3274.

Research on the possibilities of classical limit analysis is still being carried out. Recently, improved graphic oriented techniques for analysis of masonry arch and vault structures, based on the combination of the static and kinematic approaches, have been presented by Block et al. (2006). Limit analysis is currently exploited as a useful tool to analyze ancient structures (Ochsendorf, J, 2002, De Luca et al., 2004, Block, 2006). Roca et al. (2007) have produced a method for graphically-oriented analysis of reinforced masonry structures based on the same principles.

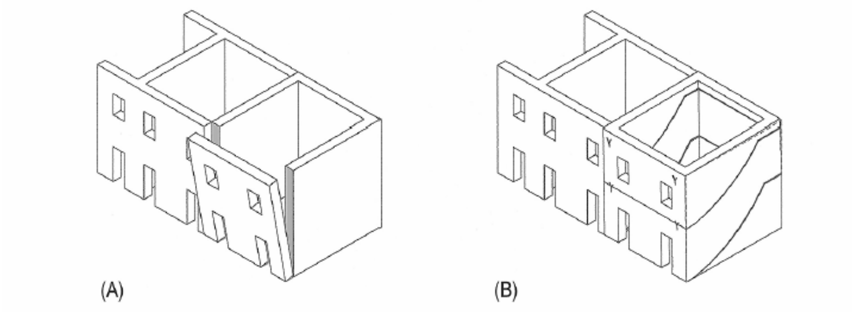


Fig. 12. Failure modes for buildings with no ties (a) and with ties anchoring the façade to lateral walls (Carocci, 2001).

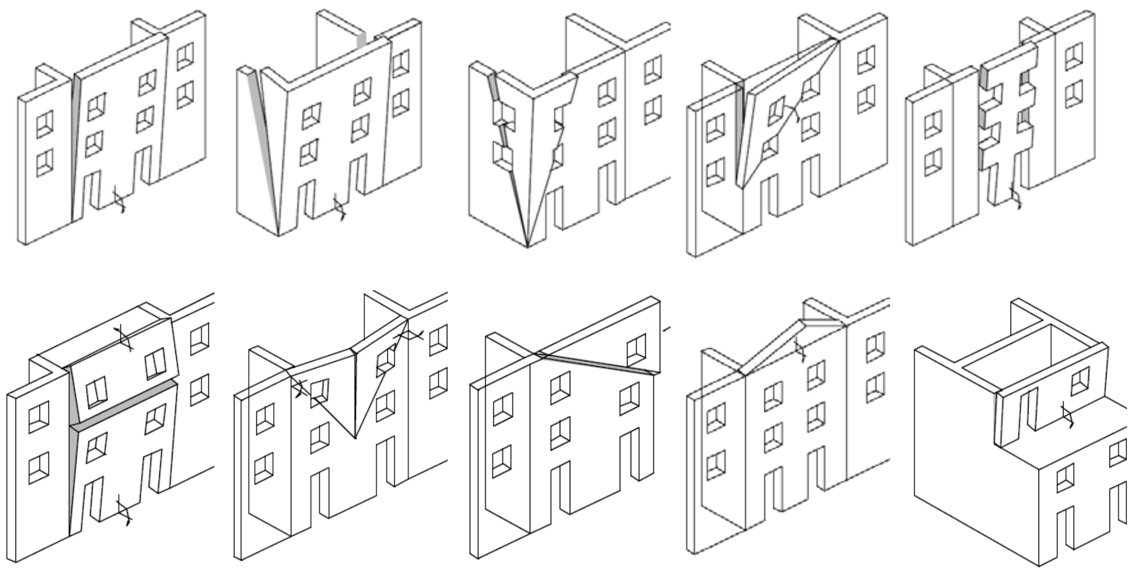


Fig.13. Failure mechanisms for buildings embedded within urban texture (based on D'Ayala and Speranza, 2002).

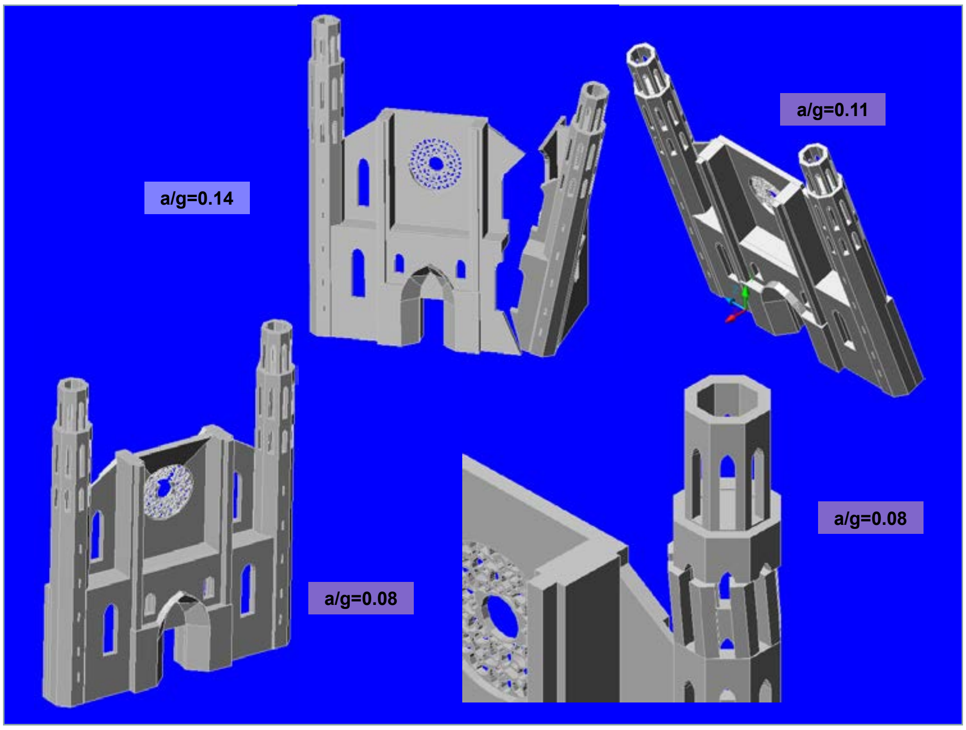


Fig. 14. Kinematic analysis applied to possible collapse models of the façade of Santa Maria del Mar church in Barcelona (Roca et al., 2009).

5. Advanced computer developments based on limit analysis

5.1 Analysis of blocky structures

Most modern computer developments based on limit analysis exploit the potential of the kinematic approach for the analysis of masonry structures composed of block assemblages. The following hypotheses are normally adopted: (1) Limit load occurs at small overall displacements. (2) Masonry has zero tensile strength. (3) Shear failure at the joints is perfectly plastic. (4) Hinging failure mode at a joint occurs for a compressive load independent from the rotation. Hypothesis (1) is true for most cases. Assumption (3) is fully supported by experimental results. In the case of masonry crushing, hypothesis (4) might be questionable, but crushing behaviour has minor importance in the response of masonry structures except for very shallow arches, pillars, towers and massive vertical structures.

The use of the aforementioned principles in combination with modern computers and advanced numerical methods has provided powerful tools for the analysis of masonry constructions. Livesley, (1978), Melbourne and Gilbert (1994) (see also Gilbert and Melbourne, 1994, and Gilbert, 2007), Baggio and Trovalusci (1998), Ferris and Tin-Loi (2001), Casapulla and d'Ayala (2001), Orduña and Lourenço (2003, 2005a, 2005b) and Gilbert et al. (2006), among others, have proposed different methods for the assessment of masonry structures by limit analysis. Modern developments afford the analysis of blocky structures for both in-plane or out-of plane loading.

Orduña and Lourenço (2003, 2005a, 2005b) have proposed a Cap Model for limit analysis for both plane or spatial structures made of rigid blocks which takes into account the non-associated flow rules and limited compressive strength of masonry. Results obtained with the model have been satisfactorily compared with available experimental results. An example concerning a simple wall subjected to seismic loads is given in Fig. 15, where the amount of horizontal forces resisted is measured as a multiplier α over the gravity forces.

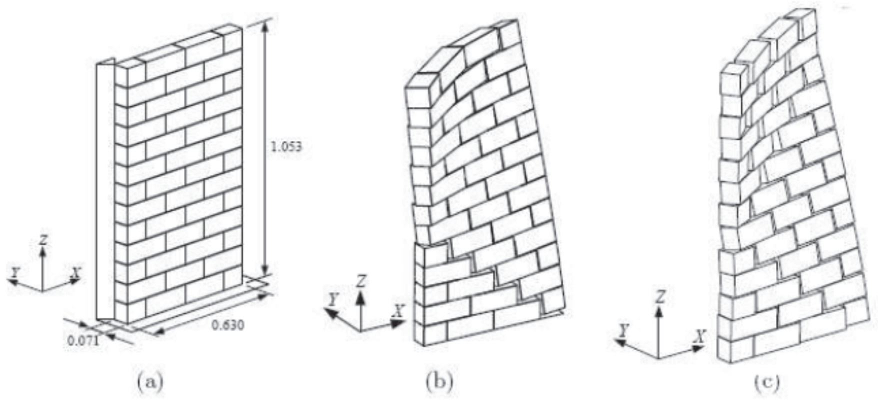


Fig. 15: Out of plane loaded wall, supported at one edge; (a) model; (b) FEM failure mechanism (α = 0.210); (c) limit analysis failure mechanism (α= 0.216) (Orduña and Lourenço, 2005b).

Limit analysis formulation for blocky structures places some difficulties regarding the treatment of the normality condition. Standard formulations adopt a simple frictional Coulomb law characterized by a friction angle Φ at the contact interfaces. Applying the normality condition (or associated flow rule) leads, in this case, to a fixed dilatancy (normal separation between blocks) characterized by an angle ψ necessarily equal to Φ, where $tan\psi$ is the ratio between normal and tangent deformation (Fig. 16). In reality, no physical condition leads to this value, real dilatancy of masonry being variable and almost null in many cases. However, the limit theorems of plasticity, previously enunciated, are only applicable when associated flow rules are adopted (i.e., when the normality condition stands). Adopting a non-associated flow rule by assuming, for instance, null dilantacy, leads to non-standard limit analysis for which the limit theorems are not strictly applicable. Considering finite compression strength (for instance, by assuming a contact region subjected to uniform yielding compression stresses) leads as well to non-standard limit analysis. As shown by Orduña and Lourenço (2005a), looking to the safest solution by minimizing the multipliers obtained by the kinematic approach (upper-bound theorem) may, for non associate flow rules, cause severe underestimation of the collapse load and incorrect failure mechanisms. Orduña and Lourenço (2005a) have overcome this problem by means of a load-path following procedure in which equilibrium and yield conditions are first applied for only constant loads. Once the constant loads have been applied, the variable ones are steadily increased. A solution for the entire set of equations is obtained for a small compressive effective stress (i.e., a reduced compression strength taking into account transverse cracking) at the interfaces and the load factor is minimized. Then, solutions are computed for successively raising compressive effective stresses until the assessed value is reached.

Gilbert et al. (2006), in turn, use a non-associative frictional joint model (a specially modified Mohr-Coulomb failure surface) which is continuously updated, within a iterative procedure, until a converged solution is obtained. The procedure provides reasonable estimates of the ultimate capacity for a wide range of problems, including entire masonry façades (Fig. 17).

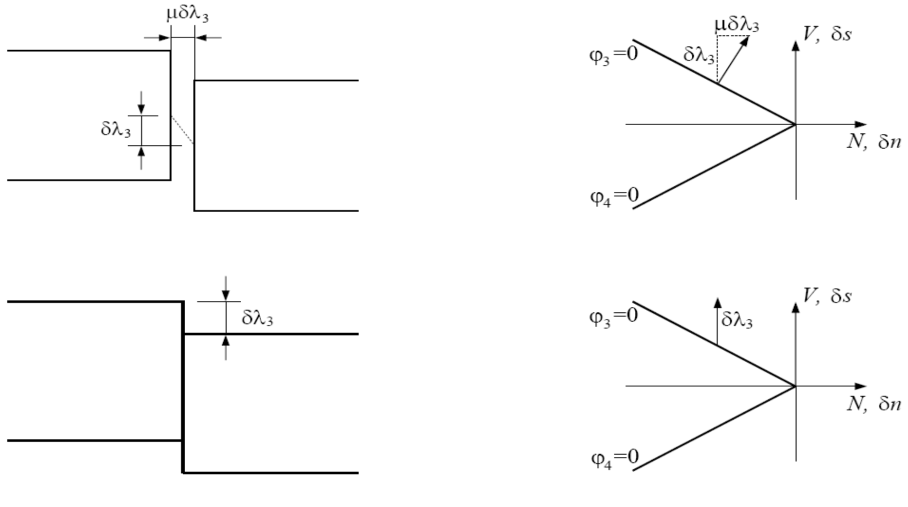


Fig. 16. Associated flow rule (top) with μ=tan*Φ*=tan*ψ,* and non-associated flow rule (below), with null dilatancy (tan *ψ=0)*. The yield criterion is represented as relationship between the normal (*N*) and shear forces (*V*). The corresponding displacements are the normal (*δn*) and tangent (*δs*) ones.

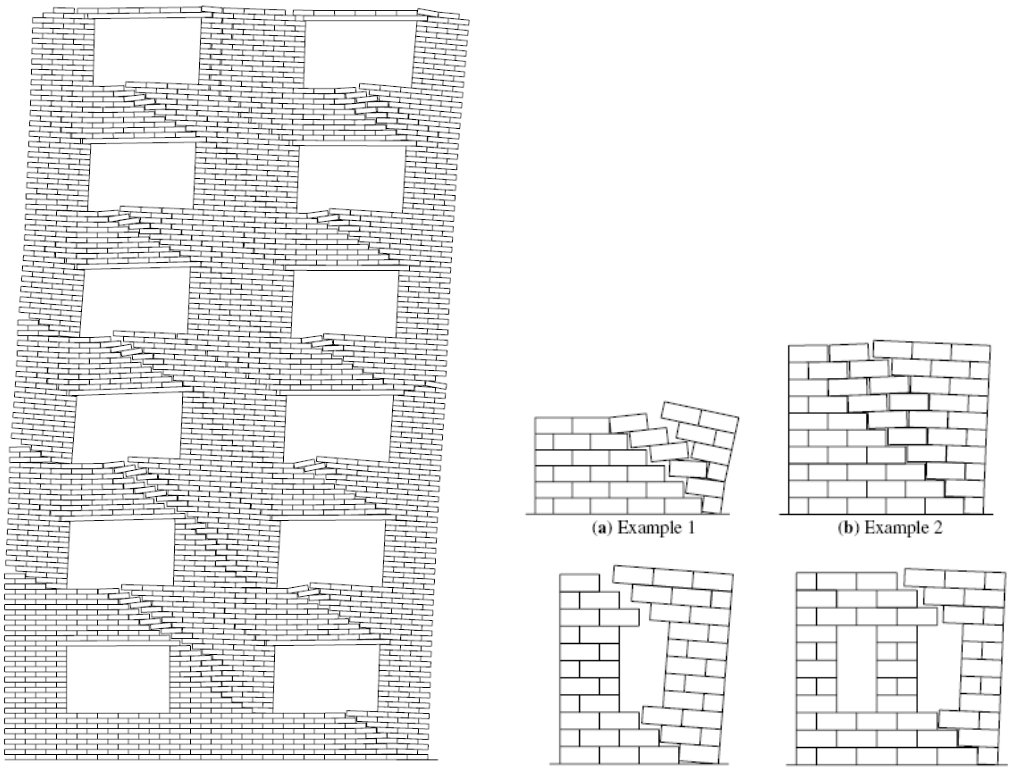


Fig. 17. Failure modes for walls subjected to vertical and horizontal loading obtained by Gilbert et al. (2006).

5.2 Limit analysis of domes, vaults and complex spatial structures

Recent proposals are exploring the potential of the lower-bound theorem (static approach) for the analysis of vaulted and 3D spatial systems. A formulation for the analysis of curved shell masonry members was presented by O'Dwyer (1999), consisting of the decomposition of the shell element into a system of arches in equilibrium. By applying the safe theorem and maximizing the ultimate load, via an optimization process, O'Dwyer establishes a procedure which allows the general application of Heyman's limit analysis to vaults and domes. Block and Ochsendorf (2007) have presented a *Thrust-Network Analysis* method for generating compression-only surfaces and spatial systems based on a duality between geometry and in-plane forces in networks. The method is applicable to the analysis of vaulted historical structures and for the design of new vaulted ones. Lucchesi et al. (2007) have generalized the static approach for arches to the case of masonry vaults through a maximum modulus eccentricities surface, a concept analogous to the line of thrust in the case of arches.

Andreu et al. (2006, 2007) have developed a computer technique for the assessment of complex masonry constructions, including 3D framed structures and shells, inspired on Gaudi's hanging models (Fig.18). Skeletal masonry constructions are modelled as 3D catenary nets composed of numerous virtual strings subjected to arbitrary loading. Based on this description, limit analysis is applied according to the static approach. Cable net solutions complying with the limit theorems of plasticity –in particular, the safe (or lower-bound) and the uniqueness theorems- are generated by means of convenient optimization techniques.

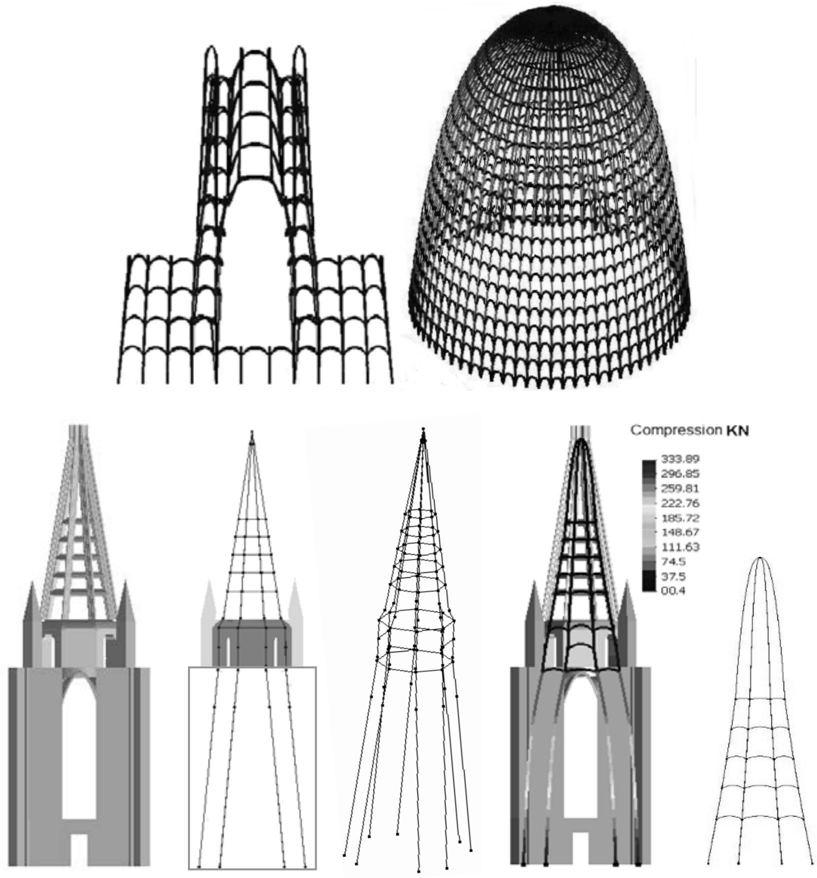


Fig 18. Examples of inverted multiple nets (top) and application to describe the equilibrium condition of a tower of the façade of Barcelona Cathedral (Andreu et al., 2006).

6. Elastic linear analysis. Possibilities and limitations.

Linear elastic analysis is commonly used in the calculation of steel and reinforced concrete structures. However, its application to masonry structures is, in principle, inadequate because it does not take into account the non-tension response and other essential features of masonry behaviour. It must be noted that, due to its very limited capacity in tension, masonry shows a complex non-linear response even at low or moderate stress levels. Moreover, simple linear elastic analysis cannot be used to simulate masonry strength responses, typically observed in arches and vaults, characterized by the development of partialized subsystems working in compression. Attempts to use linear elastic analysis to dimension arches may result in very conservative or inaccurate approaches. Linear elastic analysis is not useful, in particular, to estimate the ultimate response of masonry structures and should not be used to conclude on their strength and structural safety.

Notwithstanding, linear elastic analysis has been used, with partial success, as an auxiliary tool assisting in the diagnosis of large masonry structures. Easy availability and reduced computer costs have promoted its use, in spite of the mentioned limitations, before the development and popularization of more powerful computer applications.

Some examples are the studies of San Marco in Venice (Mola and Vitaliani, 1995), Fig. 3, the Metropolitan Cathedral of Mexico (Meli and Sánchez-Ramírez 1995), the Tower of Pisa (Macchi et al., 1993), the Colosseum of Rome (Croci, 1995) (Fig.2) and the Church of the Güell Colony in Barcelona, by Gonzalez et al. (1993), see also and Roca (1998), Fig. 19, among many other. The case of Hagia Sophia has deserved much attention and has been analyzed by different authors using similar modeling techniques (Mark et al, 1993a, 1993b, Croci et al, 1997, and others). In all these cases, the limitations of the method were counterbalanced by the very large expertise and deep insight of the analysts.

During the last years, non-linear analysis is becoming more popular thanks to larger software availability and increasing computer capacity. However, linear analysis is always performed, prior to the application of more sophisticated approaches, to allow a quick and first assessment of the adequacy of the structural models regarding the definition of meshes, the values and distribution of loads and reactions, and the likelihood of the overall results.

Meli and Peña (2004) have discussed the possibilities of elastic-linear models in providing preliminary information for the seismic study of masonry churches and the possibility of using information obtained from them for constructing more detailed models of critical parts to be then studied separately with more complex analyses.

7. Simplified modelling

7.1 Extensions of matrix calculation for linear members

The limitations of linear elastic analysis, on the one hand, and limit analysis, on the other hand, can be partly overcome by means of simple generalizations of matrix calculation of frame structures, extended with (1) Improved techniques for the description of complex geometries (curved members with variable sections…) and (2) Improved description of the material (for instance, including simple constitutive equations yet affording the consideration of cracking in tension and yielding / crushing in compression, yielding in shear).

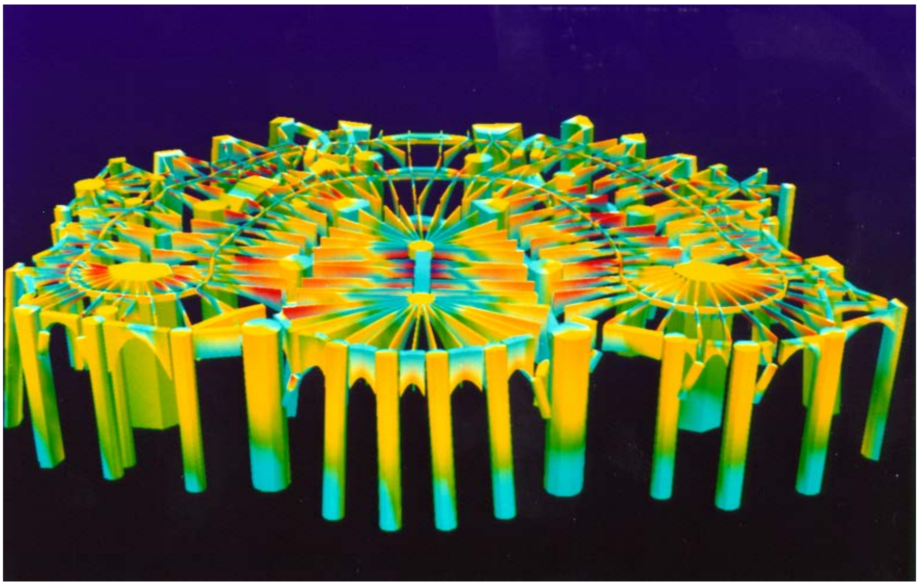


Fig. 16: Linear elastic analysis of the Crypt of the Güell Colony near Barcelona (Roca, 1998). Regions subjected to tensile normal stresses are represented in red colour.

These tools are, in principle, only applicable to 2D or 3D systems composed of linear members (namely, skeletal structures). However, there are some proposals to treat 2D members (vaults, walls) as equivalent systems composed of beams.

In fact, the application of conventional frame discretization yields inaccurate results when dealing with shear wall systems (Karantoni and Fardis, 1992). However, these results can be improved through the definition of a set of special devices to represent more realistically the shear deformation of the walls. Among some different proposals, Kwan's (1991) technique, in which the shear deformation of the walls is transferred to rigid arms connecting wall panels or walls with lintels, has shown to be very accurate for both solid or hollow walls and façades.

Molins and Roca (1998) developed a method for the analysis of masonry skeletal structures as an extension of conventional matrix calculation to systems composed of curved members with variable cross-section. The method includes a set of partial models for the description of the non-linear response of masonry taking into account cracking in tension and yielding or crushing in compression. Roca et al. (2005) extended the method to analyse 3D systems including masonry load bearing walls using Kwan's proposal for the modelling of wall systems as equivalent frames. The method has been successfully used in the assessment of façades and entire buildings.

7.2 Use of rigid and deformable macroelements

Important research efforts have been devoted to the development of computational approaches based on rigid and deformable macroelements. Each macroelement models an entire wall or masonry panel, reducing drastically the number of degrees of freedom of the structure. Brencich, Gambarotta and Lagomarsino (1998, Fig. 20) use two nodes macroelements taking into account the overturning, damage and frictional shear mechanisms experimentally observed in masonry panels. The overall response of buildings to horizontal forces superimposed to the vertical loads is obtained by assembling shear walls and flexible floor diaphragms. The former are made up of both macroelements, representative of piers and spandrels, and rigid elements, representing the undamaged parts of the walls.

More recent developments, as those due to Casolo and Peña (2007) and Chen et al. (2008), show the permanent interest of this type of simplified approaches and their ability to combine satisfactory accuracy with computer efficiency.

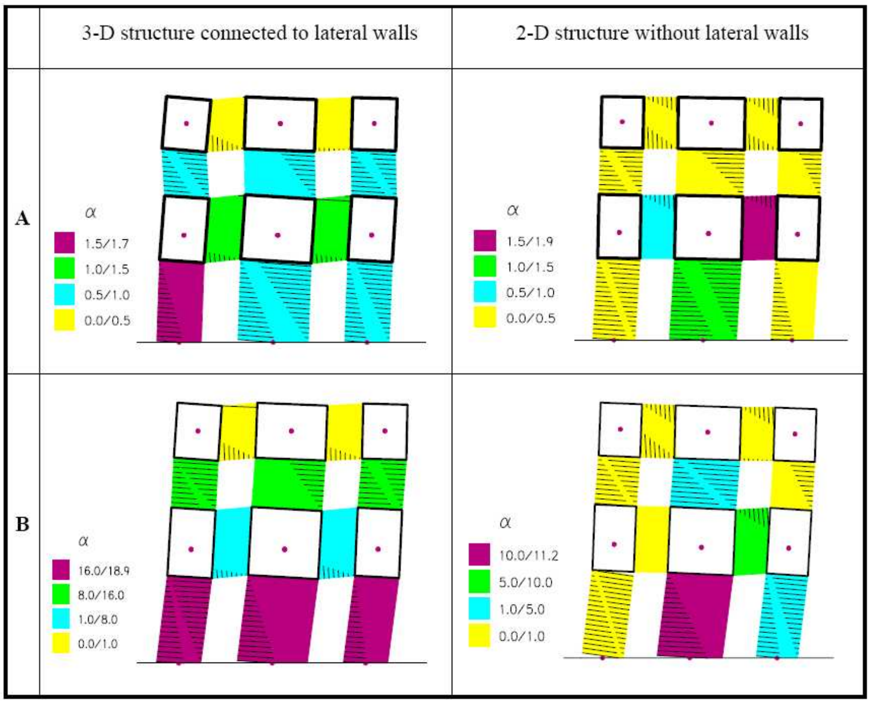


Fig. 20. Analysis with macro-blocks of a façade wall tested by Calvi and Magenes (1994). Damage distribution under monotonic loading (Brencich, Gambarotta and Lagomarsino, 1998) at peak load (A) and at the end of the monotonic load history (B).

Casolo and Peña (2007) have developed a specific rigid element approach for the in-plane dynamic analysis of masonry walls. A rigid body spring model (RBSM) has been adopted consisting of a collection of plane quadrilateral rigid elements connected to each other by two normal springs and one shear spring at each side. Specific separate hysteretic laws are assigned to the axial and shear deformation between elements. A Coulomb-like law is adopted to relate the strength of the shear springs to the vertical axial loading. The satisfactory performance of the approach has been proven by comparison with available experimental and numerical results on pier walls and façades. The technique has been also successfully applied to the study of a large real masonry construction, namely the Maniace Castle in Syracuse (Casolo and Sanjust, 2009).

8. FEM based approaches. Macro-modelling

The finite element method offers a widespread variety of possibilities concerning the description of the masonry structures within the frame of detailed non-linear analysis. An example of a pioneering application is found in Mark's et al. (1993) analysis of Hagia Sophia allowing different portions of the structure to weaken at different levels of tension.

Most of modern possibilities based on FEM fall within two main approaches referred to as macro-modelling and micro-modelling.

Macro-modelling is probably the most popular and common approach due to its lesser calculation demands. In practice-oriented analyses on large structural members or full structures, a detailed description of the interaction between units and mortar may not be necessary. In these cases, *macro-modelling,* which does not make any distinction between units and joints, may offer an adequate approach to the characterization of the structural response. The macro-modelling strategy regards the material as a fictitious homogeneous orthotropic continuum.

An appropriate relationship is established between average masonry strains and average masonry stresses. A complete macro-model must account for different tensile and compressive strengths along the material axes as well as different inelastic properties along each material axis. The continuum parameters must be determined by means of tests on specimens of sufficiently large size subjected to homogeneous states of stress. As an alternative to difficult experimental tests, it is possible to assess experimentally the individual components (or simple wallets and cores, see Benedetti *et al.* 2008) and consider the obtained data as input parameters of a numerical homogenization technique. Compared to more detailed approaches affording the description of discontinuities, macro-modelling shows significant practical advantages. In particular, FE meshes are simpler since they do not have to accurately describe the internal structure of masonry and the finite elements can have dimensions greater than the single brick units. This type of modelling is most valuable when a compromise between accuracy and efficiency is needed.

The macro-models, also termed *Continuum Mechanics finite element models*, can be related to plasticity or damage constitutive laws. An example of the former approach is the work of Lourenço (1996) and Lourenço et al. (1998) which proposed a non-linear constitutive model for in-plane loaded walls based on the plasticity theory, for which the material admissible field is bounded by a Hill-type yield criterion for compression and a Rankine-type yield criterion for tension (Fig. 21).

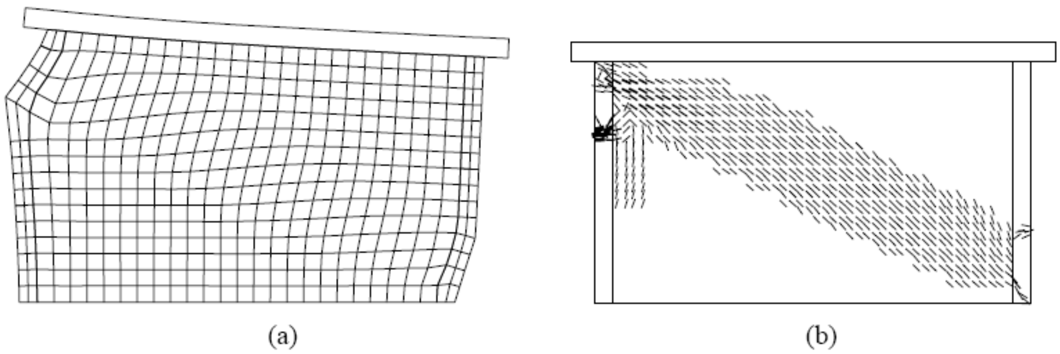


Fig. 21. Analysis of a shear wall with the plasticity model of Lourenço et al. (1998): deformed mesh (a) and cracks (b).

In the case of Continuum Damage finite element models, isotropic criteria have been usually preferred because of their mathematical simplicity and the need for only few material parameters. However, a number of orthotropic models have been also proposed. Papa’s (1996) orthotropic model consists of a unilateral damage model for masonry including a homogenization technique to keep into account the texture of brick and mortar. Berto *et al.* (2002) developed a specific damage model for orthotropic brittle materials with different elastic and inelastic properties along the two material directions. The basic assumption of the model is the acceptance of the natural axes of the masonry (i.e. the bed joints and the head joints directions) also as principal axes of the damage (Fig. 22).

The macro-models have been extensively used with the aim of analyzing the seismic response of complex masonry structures, such as arch bridges (Pelà *et al.*, 2009), historical buildings (Mallardo *et al.*, 2008), and mosques and cathedrals (Roca *et al.*, 2004, Martínez *et al.*, 2006; Murcia-Delso et al., 2009, Figs. 23 and 24).

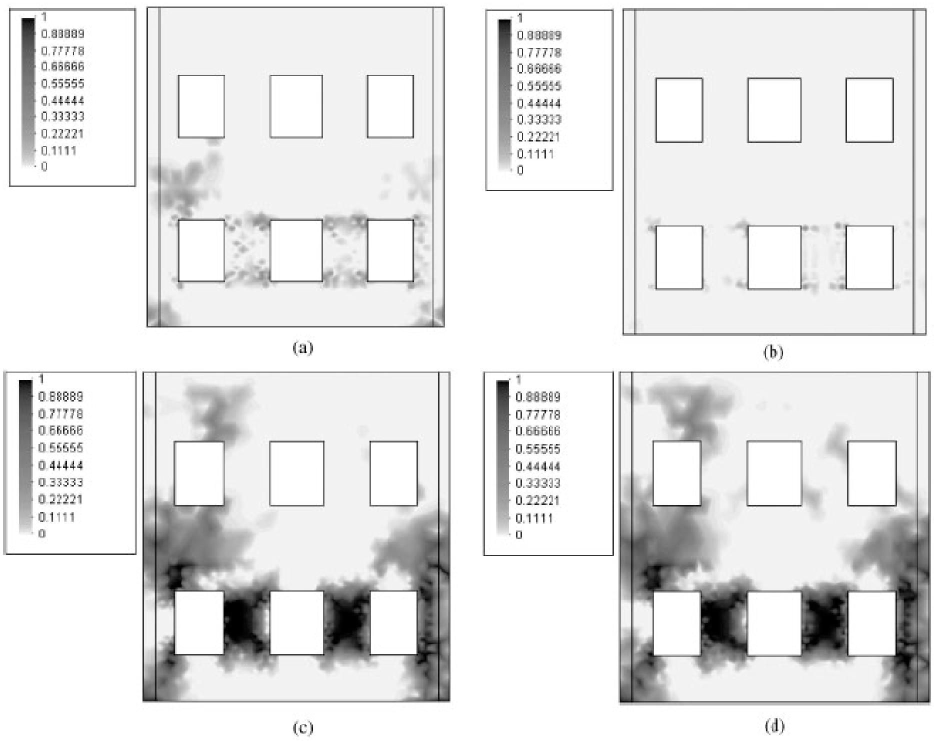


Fig. 22. Analysis of a cyclically-loaded wall with openings (from Berto et al. 2002): (a) dx- , (b) dy- , (c) dx+ and (d) dy+ numerical damage contours.

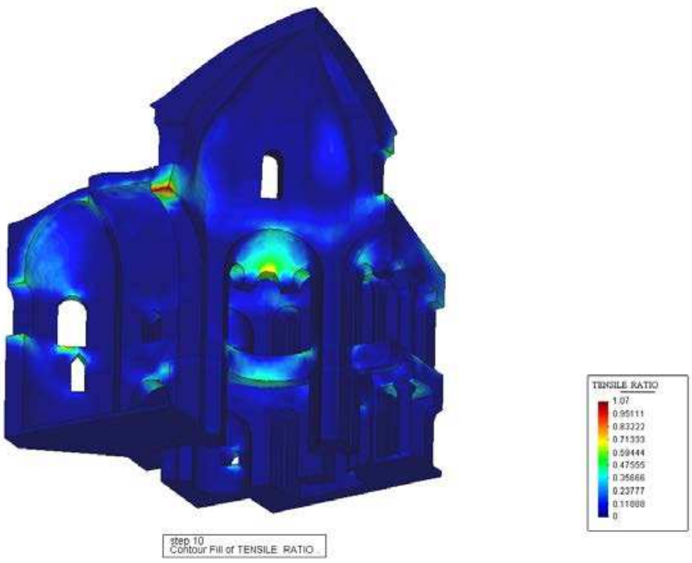


Fig. 23. Analysis of Küçük Ayasofya Mosque in Istanbul by a damage mechanics –based macro-model. Distribution of tensile damage parameter (in chromatic scale) for the structure subjected to dead loading (Roca et al. 2004).

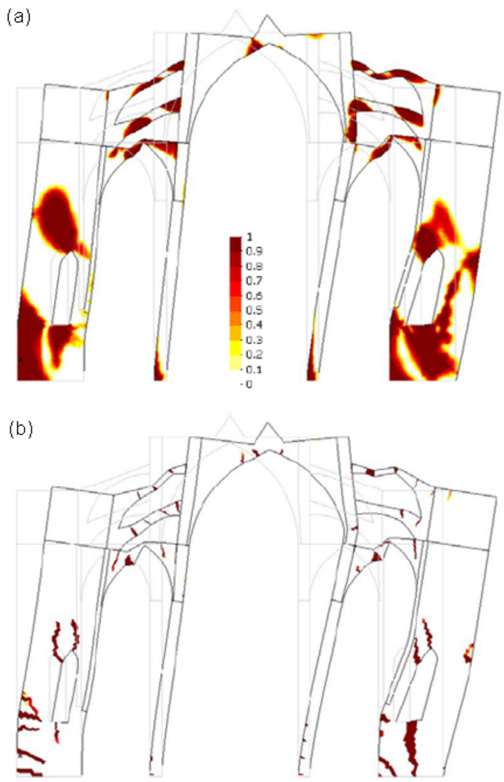


Fig. 24. Seismic analysis of Mallorca Cathedral. Smeared damage approach (a) versus localized damage approach (b), for an earthquake of 475 years of return period (Clemente et al., 2006). Both diagrams represent the tensile damage scalar parameter in chromatic scale.

A drawback of the macro-modelling approach lays in its description of damage as a smeared property spreading over a large volume of the structure. In real unreinforced masonry structures, damage appears normally localized in isolate large cracks or similar concentrated lesions. A smeared modelling of damage provides a rather unrealistic description of damage and may result in predictions either inaccurate or difficult to associate with real observations.

An interesting enhancement to smeared damage representation was recently proposed by Clemente *et al*. (2006). The method is based on the so-called smeared-crack scalar damage model, modified in such a way that it can reproduce localized individual (discrete) cracks. This is achieved by means of a local crack-tracking algorithm. The crack tracking model enables the simulation of more realistic damage distributions than the original smeared-crack model. The localized cracks predicted by the crack tracking model behave, in consistency with limit analysis, as a set o hinges developing gradually and finally leading to a full collapsing mechanism. The model has been used to analyze the response of the structure of Mallorca Cathedral under gravity and seismic forces (Fig. 24). More recently, Clemente's et al. (2006) isotropic damage has been modified to account for masonry's orthotropy by Pelà et al. (2008, see also Pelà 2009).

Very few attempts to combine non-linear constitutive equations with solid 3D element meshes have been formulated. Oñate et al., (1995), see also Hanganu (1997), proposed a 3D continuum damage model for masonry and concrete under physical and environmental effects and applied it to the study of the Domes of the San Marco's Basilica in Venice. The model was later used to study Barcelona Cathedral (Roca et al. 1998).

9. FEM based approaches. Discontinuous models

Macro-models encounter a significant limitation in their inability to simulate strong discontinuities between different blocks or parts of the masonry construction. Such discontinuities, corresponding either to physical joints or individual cracks formed later in the structure, may experience phenomena such as block separation, rotation or frictional sliding which are not easily describable by means of a FEM approach strictly based on continuum mechanics.

A possible way of overcoming these limitations consists of the inclusion within the FEM mesh of joint interface-elements to model the response of discontinuities. A pioneering application of this possibility is found in Mark's et al. (1993a) study of the buttresses of Hagia Sophia in Istambul by means of a model combining large blocks with interface elements allowing the sliding along the large blocks shaping the structure.

A more sophisticated set of discontinuous models was used by Pegon et al. (1995, 2001) to study the seismic response of the cloisters of the São Vicente de Fora Monastery in Lisbon. A full-scale model of part of the cloisters was built in laboratory and was subjected to intensive experimentation to, among other purposes, validate the models. The models included joint models to simulate the block-to-block and block-to-masonry interfaces. Elastic-perfectly-plastic and material softening joint models were used. The plastic range was defined by means of a Mohr-Coulom law characterized by a friction angle and acohesion. However, the softening model allowed the initial friction angle to decrease towards a residual value, whereas the cohesion remained constant. The study included a combination of 2D analyses involving the entire façade with more detailed 3D analysis on selected parts of the structure (Fig. 25). The material between the joints (stone or brick masonry) was described as either linear elastic or non-linear plastic homogeneous material.

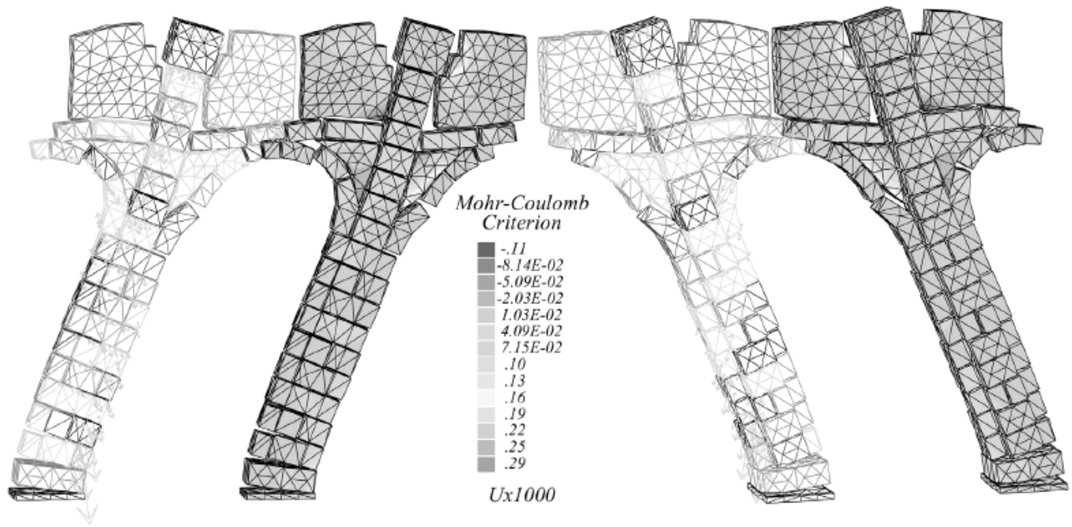


Fig. 25. 3D failure mechanism of part of the cloister of São Vicente de Fora Monastery in Lisbon as obtained by Pegon et al. (2001).

9. FEM based approaches. Micro-modelling

In the micro-modelling strategy, the different components, namely the units, mortar and the unit/mortar interface are distinctly described. The so-called *detailed* micro-models describe the units and the mortar at joints using continuum finite elements, whereas the unit-mortar interface is represented by discontinuous elements accounting for potential crack or slip planes (Fig. 26). Detailed micro-modelling is probably the more accurate tool available to simulate the real behaviour of masonry. It is particularly adequate to describe the local response of the material. Elastic and inelastic properties of both unit and mortar can be realistically taken into account.

The detailed macro-modelling strategy leads to very accurate results, but requires an intensive computational effort. This drawback is partially overcome by the *simplified* micro-models (Lotfi and Shing, 1994; Tzamtzis, 1994; Lourenço and Rots, 1997, Fig. 27; Gambarotta and Lagomarsino, 1997, Fig. 28; Sutcliffe et al., 2001), where expanded units, represented by continuum elements, are used to model both units and mortar material, while the behaviour of the mortar joints and unit-mortar interfaces is lumped to the discontinuous elements (Fig. 26). Masonry is thus considered as a set of elastic blocks bonded by potential fracture/slip lines at the joints.

The micro-modelling approaches are suitable for small structural elements with particular interest in strongly heterogeneous states of stress and strain. The primary aim is to closely represent masonry based on the knowledge of the properties of each constituent and the interface. The necessary experimental data must be obtained from laboratory tests on the constituents and small masonry samples. Nevertheless, the high level of refinement required means an intensive computational effort (i.e. great number of degrees of freedom of the numerical model), which limits micro-models applicability to the analysis of small elements (as laboratory specimens) or small structural details.

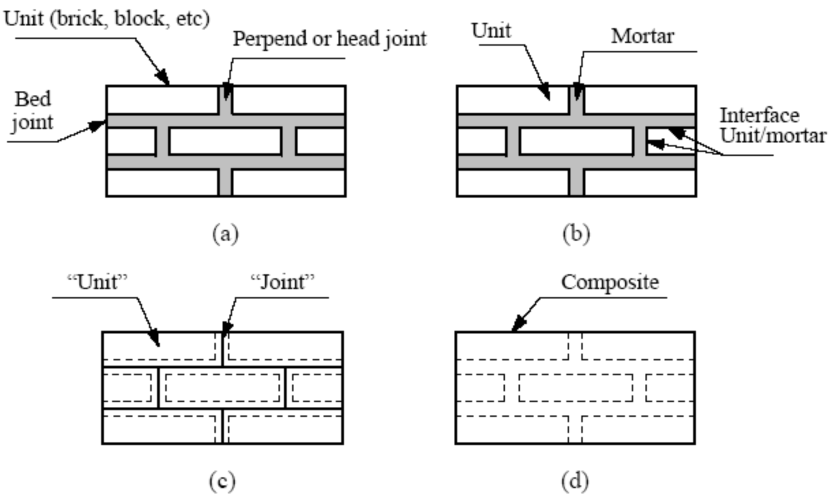
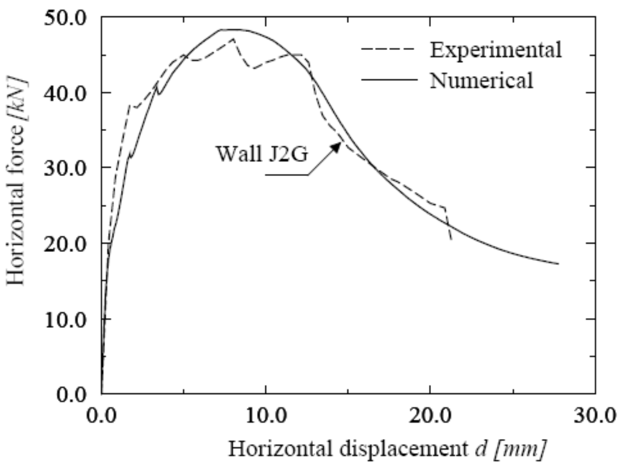


Fig. 26. Modelling strategies for masonry structures (Lourenço, 1996): masonry sample (a); detailed (b) and simplified (c) micro-modelling; macro-modelling (d).



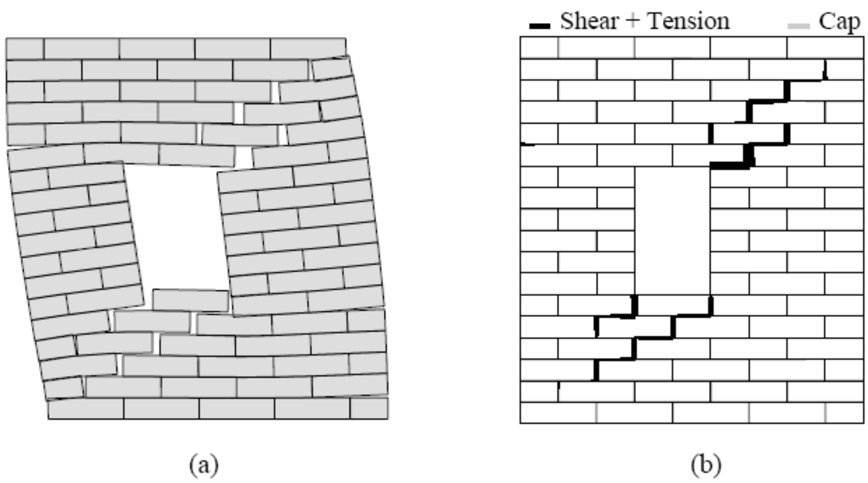


Fig. 27. Micro-modelling of masonry shear walls (Lourenço, 1996). Top: Load-displacement diagram. Bottom: Results of the analysis at a lateral displacement of 2.0 mm: (a) deformed mesh; (b) damange.

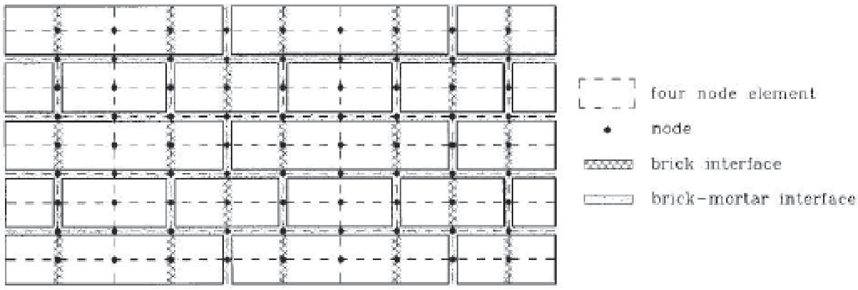


Fig. 28. Micro-modelling of masonry, from Gambarotta and Lagomarsino (1997).

10. Homogenization

Midway between micro-modelling and macro-modelling stands the so-called *homogenized modelling*. If the structure is composed by a finite repetition of an elementary cell, masonry is seen as a continuum whose constitutive relations are derived from the characteristics of its individual components, namely blocks and mortar, and from the geometry of the elementary cell. Most of the methods of homogenization simplify the geometry of the basic unit with a 2 step introduction of vertical and horizontal joints and thus without taking into account the regular offset of vertical mortar joints. However, this kind of approach results in significant errors when applied to non-linear analysis.

Micromechanical homogenization, derived independently by van der Pluijm (1999), Lopez et al. (1999) and Zucchini and Lourenço (2002), is based on the detailed finite element analysis of the elementary cell and overcomes the approximations introduced by the 2 steps simplified method. Recent advances in terms of sophisticated analysis homogenization tools are discussed in Lourenço et al. (2007) and include the polynomial stress field expansion approach of Milani et al. (2006a) and the mesoscopic approach of Massart et al. (2004), Calderini and Lagomarsino (2006), and Shieh-Beygia and Pietruszczak (2008).

A micro-mechanical model for the homogenized limit analysis of in-plane loaded masonry has been proposed by Milani et al. (2006a, 2006b, 2006c, 2007). It is developed with the aim to obtain the homogenised failure surfaces for masonry. The strength domains are implemented in finite element limit analysis codes and numerically treated both with a lower and an upper bound approach (Fig. 29). The main advantages of this method, compared with classical micro-modelling, are the following: (1) The finite element mesh does not have to reproduce the exact pattern of the masonry units nor it has to be so fine. The structure can be meshed automatically. (2) Once the homogeneous properties have been calculated from the micro-mechanical model, standard finite element method can be used to perform the analysis avoiding the complications introduced by elements interfaces.

The micro-mechanical approach has been successfully applied to both linear and non-linear problems. In a first step, Zucchini and Lourenço (2002) introduced homogenization in the elastic field by deriving the mechanical properties of masonry from a suitable micro-mechanical model taking into account the staggered alignment of the units. The homogenisation model has been subsequently extended to non-linear problems in the case of a masonry cell failure under tensile loading parallel to the bed joint, Zucchini and Lourenço (2004), or under compressive loading perpendicular to the bed joint, Zucchini and Lourenço (2007). These results are accomplished by coupling the elastic micro-mechanical model with a damage model in tension and a plasticity model in compression. To extend the model to mixed loading conditions Zucchini and Lourenço (2009) used a full periodic cell with an antisymmetric deformation mechanism. The internal structure of the cell is represented by five different components, namely units, two antisymmetric bed joints, head joints and cross joints

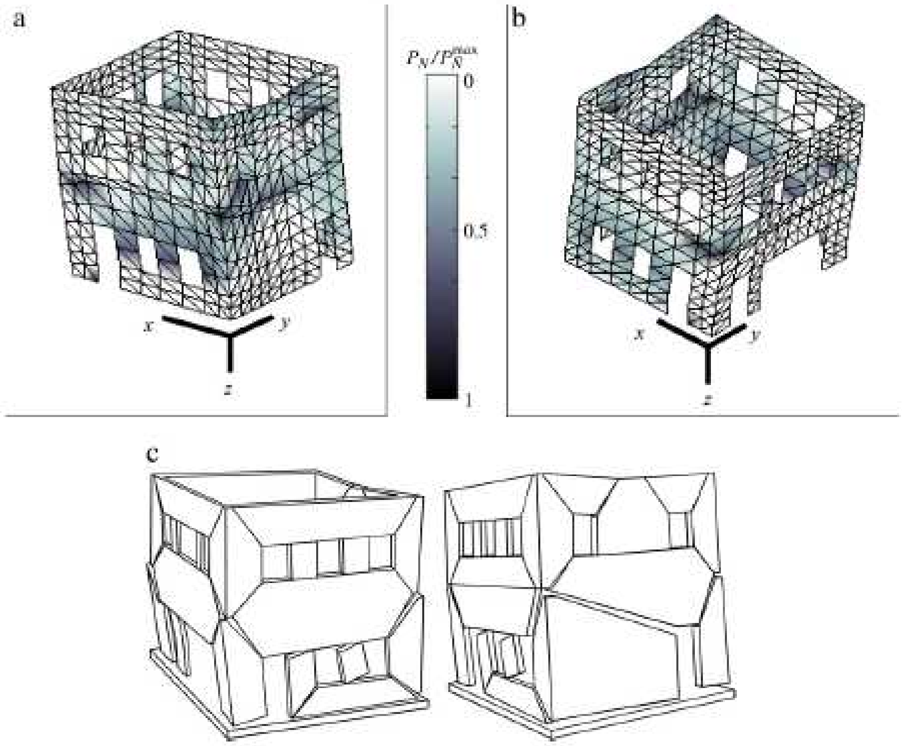


Fig.29. 3D homogenized limit analysis of a masonry building (Milani et al. 2007).

11 Non-linear FEM analysis of vaulted structures

Few non-linear developments have been produced aiming to the analysis of vaulted structures. Domes and vaults pose significant difficulties due to their curved, two-dimensional and spatial character. In fact, most of the applications for non-linear analysis of masonry structures so far mentioned are oriented to two-dimensional planar problems.

A pioneering work is found in the studies by Oñate et al. (1995) of the domes of San Marco Basilica in Venice. The set of vaults was modelled by means of a continuum damage model for masonry and concrete under mechanical and other (physical, chemical, biological) deterioration effects. The analysis attained a characterization of the safety condition of the system of vaults (Fig. 30).

Croci et al. (1998a) carried out a finite element analysis of the Cathedral of Sta. María, in Vitoria, Spain. The analysis, applied to the main transverse sections of the building and to the nave vaults, followed an incremental strategy to account for cracking due to tension or shear stresses, as well as the equilibrium second-order effects. Similar analyses were also used for the study of the collapse of Beauvais Cathedral (Croci et al., 1998b) and the effects of the earthquake of September 1997 on the Basilica of Assisi (Croci, 1998).

Barthel (1993) elaborated very detailed finite element models to analyze Gothic cross-vaults. The models were used in combination with partial constitutive models enabling the simulation of masonry cracking as well as sliding between arch ring joints.

Cauvin and Stagnitto (1993, 1995) carried out studies on Gothic cross vaults using both limit analysis and FEM non-linear analysis. Their method was successfully applied to the study of the central nave of Reims Cathedral.

Lourenço (1997) proposed a formulation for the study of masonry spatial and curved shells. To our knowledge, it is the only one based on the micro-modelling technique. It includes constitutive equations, stemming from plasticity, to simulate the response of the material in combination with joint elements to describe the sliding of blocks.

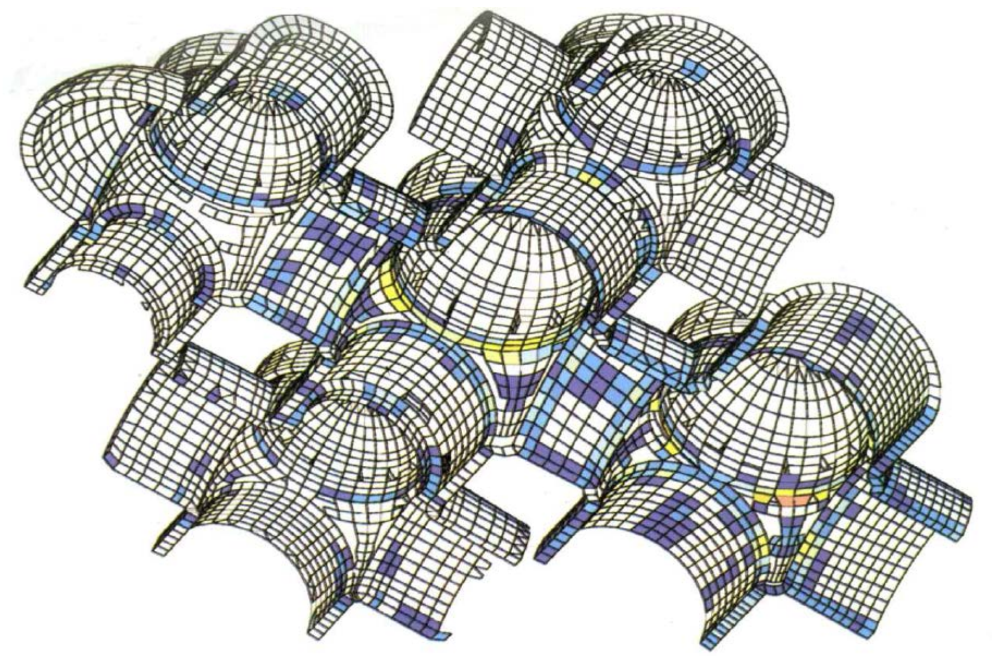


Fig. 30. Study of Saint Marco's domes in Venice by a continuum damage model (Oñate et al.,1995).

12. Discrete element method

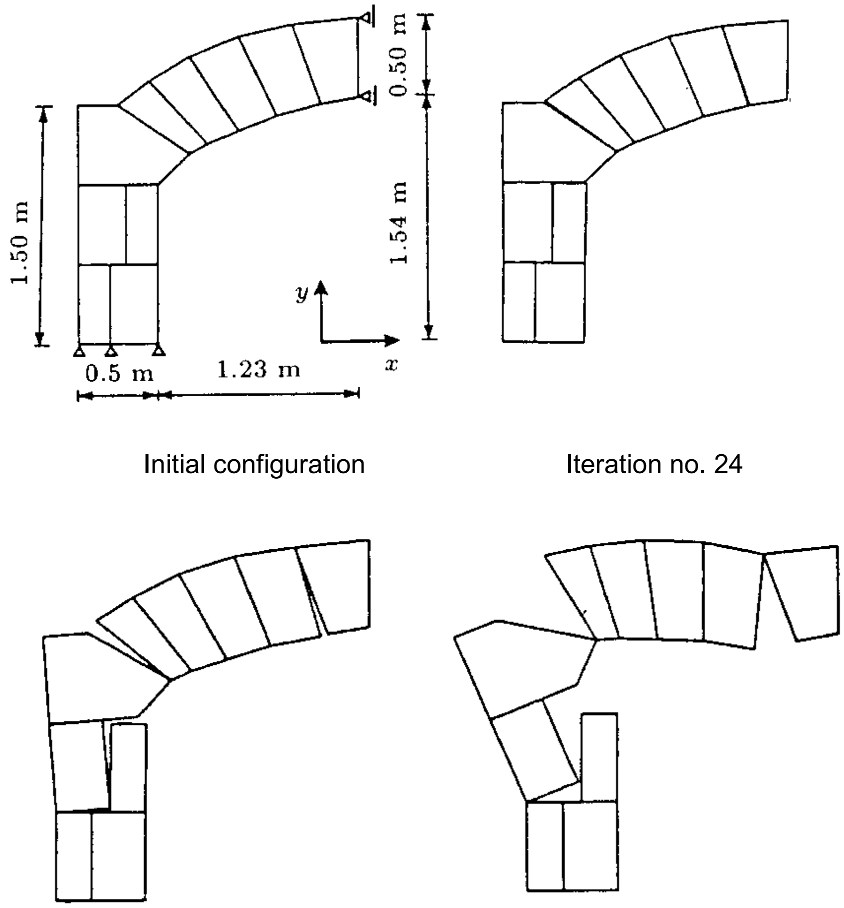
The Discrete element method (DEM) is characterized by the modeling of the material as an assemblage of distinct blocks interacting along the boundaries. According to its pioneer proposers, Cundall and Hart (1971), the name "discrete element" applies to a computer approach only if (1) it allows finite displacements and rotations of discrete bodies, including the complete detachment and (2) it can recognize new contacts between blocks automatically as the calculation progresses.

The formulation, initially oriented to the study of jointed rock, was later extended to other engineering applications also requiring a detailed study of contact between blocks or particles, such as soils and other granular materials (Ghaboussi J., Barbosa R., 1990). Finally, it has also been applied to the modeling of masonry structures (Pagnoni, 1994; Lemos, 1998; Sincraian, 2001). The common idea in the different applications of the discrete element method to masonry is the idealization of the material as a discontinuum where joints are modeled as contact surfaces between different blocks. This approach affords the modeling of various sources of non linear beahaviour, including large displacements, and suits the study of failures in both the quasi static and dynamic ranges.

Distinct element methods, discrete-finite elements and discontinuous deformation analysis are different formulations of the discrete element method with important applications to masonry structures.

Distinct element methods are direct derivations of the first work by Cundall and Hart. They involve soft contact formulations where a normal interpenetration is needed to recognize contact between two different bodies. The main features of these methods are: (1) No restriction of block shapes and no limitation to the magnitudes of translational and rotational displacements. The approximation that all the deformations occur at the surfaces of blocks is made. It is also assumed that forces arise only at contacts between a corner and an edge. (2) Forces arise due to deformation. A change in displacement results in a change in force which is added to the existing force stored for the contact. (3) Accelerations are computed from the forces and moments for each block. The accelerations are further integrated into velocity and displacements. (4) Contact updating is performed when the sum of the displacements of all of the elements has exceeded a certain value. To increase the efficiency, only blocks within a certain distance range are checked for new contacts.

Discrete-finite element methods recollect different attempts of combining FEM with multi-body dynamics. Munjiza et al. (1995) developed a method for the simulation of fracturing problems considering deformable blocks that may split and separate during the analysis. Mamaghani et al. (1999) used a fixed contact system with a small deformation framework and finite deformations concentrated in contact elements. Contacts, discontinuities and interfaces were considered as bands with a finite thickness. The contact element was a two-nodded element having normal and shear stiffnesses. The method was applied to the stability analysis of different masonry structures. The failure mode of a masonry arch is shown in Fig. 31.



Initial configuration

Iteration no. 24

Iteration no. 27

Iteration no. 30

Fig. 31. Failure mode of a masonry arch (Mamaghani et al., 1999).

The discontinuous derformation analysis (DDA) is a 2D method developed by Shi and Goodman (1988) for rock engineering analysis subsequently applied to masonry (Ma et al. 1996). Blocks are considered deformable but with a uniform strain and stress distribution. Contact is considered rigid and no interpenetration is permitted. This condition is enforced numerically by an iterative procedure at each time step.

The natural field of application of DEMs is composed by structures formed by regurarly shaped masonry or stone blocks. Rocking motion of stone blocks (Peña et al. 2007), static and dynamic analysis of load bearing walls (Pagnoni 1994, Baggio and Trovalusci 1995, Schlegel and Rautenstrauch 2004), stone bridges (Lemos 1995, Bicanic et al 2001), columns and architrave (Papastamatiou and Psycharis 1993, Psycharis et al 2003, Fig. 33), arch and pillar (Pagnoni 1994, Pagnoni and Vanzi 1995, Lemos 1998) are typical examples of DEM analysis. The analysis of complex structures is still a controversial topic in DEM. Computational viability of analysis may limit severily the number of block elements that can be included in a model. Models prepared to simulate the response of real structures may result in too coarse or unrealistic discretizations or 2D, and specially, 3D real masonry structures.

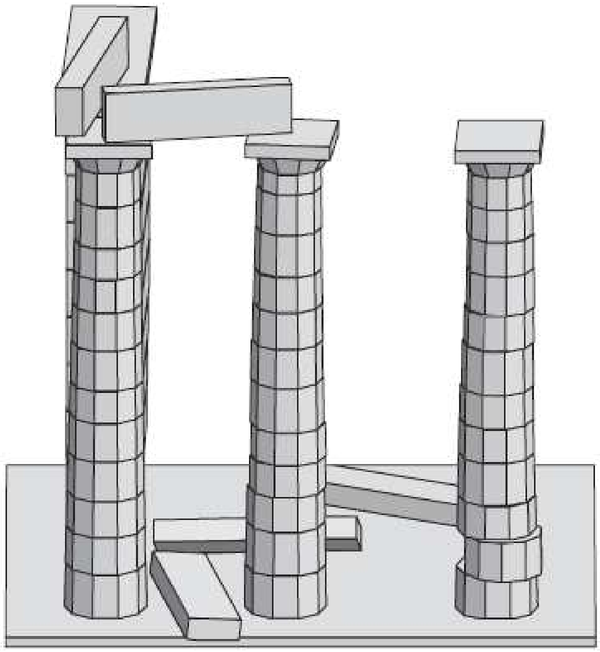


Fig. 32. Final position of the column-architrave model of the Parthenon Pronaos, without reinforcement, for a simulated earthquake (Psycharis et al., 2003).

13. Conclusions

In spite of the important advances experienced by structural analysis methods, the study of masonry historical structures is still a challenging activity due to the significant difficulties encountered in the description of their complex geometry, materials, morphology (member composition and connections), and present condition, including damage and alterations. In particular, the modelling of the mechanical behaviour of masonry is very demanding due to its fragile nature in tension and composite character at the macro-scale.

A wide set of possibilities have been developed to describe masonry structures to different levels of accuracy, from rough (but useful) engineering approaches to very detailed modelling taking into account the distinct response of the individual components. A very significant effort is undertaken at present to produce homogeneization criteria allowing a better interconnection between the micro- and macro- analysis levels.

Detailed methods, such as micro-models, are still requiring high computer effort and, in practice, can only be used to analyze small individual members such as solid or hollow walls. Moreover, the sophisticated methods often require complex material properties which can only be determined through costly and sophisticated laboratory experiments. In the case of the study of real buildings, such properties are not normally available and need to be indirectly estimated based on more common and available evidence. However, inadequate assumptions on these properties may compromise the real gain in accuracy provided by sophisticate method and even lead to inadequate results.

With further developments in computer technology and numerical methods, the analysis of entire complex historical structures (including for instance, Gothic cathedrals) using very accurate approaches, may become possible in the near future. Further developments will make very demanding analyses, such as the non-linear time-domain dynamic one, possible even if used in combination with complex structural and material models. However, many of the difficulties mentioned, as those related to the adequate survey and description of the structure, including material features and damage, will still compromise the accuracy and realism of the numerical predictions. In this context, the judgment of the analyst is essential in order to conclude on the acceptability of the results.

Sufficient validation or calibration of numerical models, based on the comparison with empirical information, will be always necessary to grant the reliability of the numerical models and their capacity to predict on the structural response and safety. In the case of historical structures, this empirical information can be obtained from historical investigation on the past performance, inspection and monitoring (Icomos/Iscarsah Comittee, 2005). In the study of a historical structure, structural analysis is a key activity to be developed in combination with other complementary, but also important activities oriented to both gathering the input information and allowing later calibration or validation of results.

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